MATH2050C Assignment 8

Deadline: March 19 , 2024.

Hand in: 4.1 no. 11b, 12d, 15; 4.2 no. 1c, 2b, 11d, 12; Supp. Ex. no. 3.

Section 4.1 no. 7, 8, 9bd, 10b, 11b, 12bd, 15.

Section 4.2 no. 1bc, 2bd, 11cd, 12.

Supplementary Problems

1. Prove by the Limit Theorem (see next page) that $\lim_{x\to c} p(x) = p(c)$ for any polynomial p and real number c.

2. Let f be a function A and c a cluster point of A. Show that $\lim_{x\to c} |f(x)| = |L|$ whenever $\lim_{x\to c} f(x) = L$.

3. Let f be a non-negative function on A and c a cluster point of A. Suppose that $\lim_{x\to c} f(x) = L$ for some L. Show that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$. Suggestion: Consider $L > 0$ and $L = 0$ separately.

See next page

The Limit Theorem for Functions

8.1 Limit Theorem Let c be a cluster point of A and f, g are functions on A satisfying $f(x) \to L, g(x) \to M$ as $x \to c$ respectively. Then

1. $\lim_{x\to c}(\alpha f + \beta g) = \alpha L + \beta M$.
2. $\lim_{x\to c}(fg)(x) = LM$.
3. $\lim_{x\to c} \left(\dfrac{f}{g}\right)(x) = \dfrac{L}{M}$ provided $M \neq 0$.

By induction, (1) and (2) of this theorem also hold for finitely many functions.

8.2 Limit Theorem Let c be a cluster point of A and $f_k, k = 1, \cdots, n$ are functions on A satisfying $f_k(x) \to L_k$ as $x \to c$. Then

1. $\lim_{x\to c} \sum_{k=1}^n \alpha_k f_k(x) = \sum_{k=1}^n \alpha_k L_k$.
2. $\lim_{x\to c}(f_1 f_2 \cdots f_n)(x) = L_1 L_2 \cdots L_n$.

In our textbook this theorem is proved by the Sequential Criterion. In class we proved it by using the ε-δ definition. Here we repeat it for the product rule. Indeed, we have

$$\begin{aligned} |(fg)(x) - LM| &= |f(x)g(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \ . \end{aligned}$$

As $g(x) \to M$, for $\varepsilon = 1$, there is some δ_1 such that $|g(x) - M| < 1$ for $x \in A, 0 < |x - c| < \delta_1$. So $|g(x)| \leq |M| + 1$ there. We have

$$|(fg)(x) - LM| \leq (|M| + 1)|f(x) - L| + |L||g(x) - M| \ ,$$

whenever $0 < |x - c| < \delta_1$. Now given $\varepsilon > 0$, as $f(x) \to L$ and $g(x) \to M$, there are δ_2, δ_3 such that $|f(x) - L| < \varepsilon/2(|M| + 1)$ for $0 < |x - c| < \delta_2$ and $|g(x) - M| < \varepsilon/2(|L| + 1)$ for $0 < |x - c| < \delta_3$. It follows that for $x, 0 < |x - c| < \delta$ where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$,

$$|(fg)(x) - LM| \leq (|M| + 1)\varepsilon/2(|M| + 1) + |L|\varepsilon/2(|L| + 1) < \varepsilon \ ,$$

done.

8.3 Sequential Criterion The following statements are equivalent:

(a) $\lim_{x\to c} f(x) = L$;

(b) For any sequence $\{x_n\}, x_n \neq c, x_n \to c, f(x_n) \to L$ as $n \to \infty$.

This criterion has three consequences. First we use it to show some common limits exist.

Example 1 Let p be a polynomial. For $c \in \mathbb{R}$, $\lim_{x\to c} p(x) = p(c)$. A polynomial is well-defined everywhere on the real line. It is of the form $a_0 + a_1 x + \cdots + a_n x^n$ for some n. It was shown in Chapter 3 that $\lim_{n\to\infty} p(x_n) = p(c)$ for any sequence $x_n \to c$. By the Sequential Criterion

$\lim_{x\to c} p(x) = p(c)$.

Example 2 Let p, q be two polynomials. For c satisfying $q(c) \neq 0$, $\lim_{x\to c} p(x)/q(x) = p(c)/q(c)$. This conclusion comes from Example 1, Sequential Criterion and the quotient rule for sequences.

Example 3 In Chapter 3 it was shown that the function $x^{p/q}, p, q \in \mathbb{N}$, is well-defined for $x \in [0, \infty)$. And for any sequence $x_n \to c \in [0, \infty)$, $x_n^{p/q} \to c^{p/q}$. Immediately it follows from the Sequential Criterion that $\lim_{x\to c} x^{p/q} = c^{p/q}$.

The second consequence is the Divergence Criteria which follow directly from the Sequential Criterion.

8.4 Divergence Criteria $\lim_{x\to c} f(x)$ does not exist in either one of the following two cases:

(a) There is $x_n \in A, x_n \neq c, x_n \to c$, such that $\{f(x_n)\}$ is unbounded;

(b) There are $x_n, y_n \in A$ not equal to c and $x_n, y_n \to c$ such that $f(x_n) \to L_1, f(y_n) \to L_2$ with $L_1 \neq L_2$.

Example 4 $\lim_{x\to 0} 1/x^p, p \in \mathbb{N}$, does not exist. Consider the sequence $x_n = 1/n \to 0$, we have $1/x_n^p = n^p \to \infty$. By the Divergence Criterion (a) this limit does not exist.

Example 5 $\lim_{x\to 0} \sin 1/x$ does not exist. Consider two sequences $x_n = 1/2\pi n$ and $y_n = 1/(2\pi n + \pi/2)$. Then $\sin 1/x_n = \sin 2\pi n = 0$ and $\sin 1/y_n = \sin(2\pi n + \pi/2) = 1$ for all n. Hence $L_1 = 0$ and $L_2 = 1$. By Divergence Criterion (b), the limit does not exist.

The third consequence of the Sequential Criterion is the Squeeze Theorem.

8.5 Squeeze Theorem Suppose that $f(x) \leq g(x) \leq h(x),\ x \in A$, and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L\ .$$

Then $\lim_{x\to c} g(x) = L$.

This theorem follows from the corresponding theorem for sequences and the Sequential Criterion.

Example 6 $\lim_{x\to 0} \sin x = 0$. Using the estimate $0 \leq \sin x \leq x$ for $x \in [0, 1]$ and the fact that the sine function is odd, $-|x| \leq \sin x \leq |x|, x \in [-1, 1]$. A direct application of the Squeeze Theorem gives the desired limit.

Example 7 $\lim_{x\to 0} \sin x/x = 1$. This follows readily from the estimate $x - x^3/6 \leq \sin x \leq x, x \in [0, 1]$ (see below). Derive both sides by x, we have $1 - x/6 \leq \sin x/x \leq 1$. Since $\sin x/x$ is even, this estimate holds on $[-1, 0) \cup (0, 1]$. By Squeeze Theorem $\lim_{x\to 0} \sin x/x = 1$.

Trigonometric Functions

A rigorous definition of the sine and cosine functions will not be introduced until Chapter 8 of our textbook. However, in order to have more diverse examples we are obliged to use them. Here we list some basic facts concerning these functions which are used in the subsequent development.

1. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ , \ x \in \mathbb{R} \ ,$$

2. The sine and cosine functions are 2π-periodic,

3. $\sin^2 + \cos^2 x = 1, \ x \in \mathbb{R}$,

4. $\sin(x+y) = \sin x \cos y + \sin y \cos x$, and $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

We derive from (1) the estimate

$$x - x^3/3! \le \sin x \le x, \ x \in [0,1] \ .$$

Indeed,

$$\sin x = x - \left[\left(\frac{x^3}{3!} - \frac{x^5}{5!} \right) + \left(\frac{x^7}{7!} - \frac{x^9}{9!} \right) + \cdots \right] \ .$$

Since each term $x^{2n+1}/(2n+1)! - x^{2n+3}/(2n+3)!$ is positive when $x \in [0,1]$, $\sin x \le x$. Similarly, one can prove the other inequality.